Prof. R. Starr
Mr. Troy Kravitz

Lecture Notes, Lectures 17, 18

Introducing Pareto Efficiency and Separation Theorems

<u>Fundamental Theorems of Welfare Economics</u>

First Fundamental Theorem of Welfare Economics: Every competitive equilibrium allocation is Pareto efficient.

Second Fundamental Theorem of Welfare Economics (subject to boundary conditions): Let the economy fulfill P.I (or P.V) and C.VI (or C.VII) --- that is, let preferences and technologies be convex. Then for any Pareto efficient allocation (x^{oi}, y^{oj}), there is $p \in P$, so that the allocation (x^{oi}, y^{oj}) is a competitive equilibrium at prices p, subject to a redistribution of endowment.

12.1 Pareto Efficiency

Definition: An allocation $x^i,\ i \in H$, is attainable if there is $y^j \in Y^j,\ j \in F$, so that $0 \le \sum_{i\in H} x^i \le \sum_{j\in F} y^j + \sum_{i\in H} r^i$. (The inequalities hold co-ordinatewise.)

Definition: Consider two assignments of bundles to consumers, v^i, w^i , $i \in H$. v^i is said to be Pareto superior to w^i if for each $i \in H$, $u^i(v^i) \ge u^i(w^i)$ and for some $h \in H$, $u^h(v^h) \ge u^h(w^h)$.

Definition: An attainable assignment of bundles to consumers, $w^i,\ i \in H$, is said to be Pareto efficient (or Pareto optimal) if there is no other attainable assignment v^i so that v^i is Pareto superior to w^i.

Definition: $< p^0, x^{0i}, y^{0j} >,\ p^0 \in R^N_+,\ i \in H,\ j \in F$, $x^{0i} \in R^N$, $y^{0j} \in R^N$, is said to be a competitive equilibrium if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{oj} \ge p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$

(ii) $x^{0i} \in X^i,\ p^0 \cdot x^{0i} \le M^i(p^0) = p^0 \cdot r^i + \sum_{j\in F} \alpha^{ij} p^0 \cdot y^{0j}$ and $u^i(x^{0i}) \ge u^i(x)$ for all $x \in X^i$ with $p^0 \cdot x \le M^i(p^0)$ for all $i \in H$, and

(iii) $0 \ge \sum_{i\in H} x^{0i} - \sum_{j\in F} y^{0j} - \sum_{i\in H} r^i$ with $p^0_k = 0$ for co-ordinates k so that the strict inequality holds.

This definition is sufficiently general to include the equilibrium developed in Theorem 7.1.

Economics 113, UCSD — Prof. R. Starr
Spring 2009 — Mr. Troy Kravitz

12.2 First Fundamental Theorem of Welfare Economics (1FTWE)

Theorem 12.1 (First Fundamental Theorem of Welfare Economics): Assume C.I, C.II, C.IV, C.V. Let $p^0 \in R^N_+$ be a competitive equilibrium price vector of the economy. Let w^{0i}, $i \in H$, be the associated individual consumption bundles, y^{oj}, $j \in F$, be the associated firm supply vectors. Then w^{0i} is Pareto efficient.

Proof: $u^i(w^{0i}) \geq u^i(x)$, for all x so that $p^0 \bullet x \leq M^i(p^0)$, for all $i \in H$.

- If $u^i(x)>u^i(w^{0i})$, for typical $i \in H$, then $p^0 \cdot x > p^0 \cdot w^{0i}$.
- $p^0 \cdot y > p^0 \cdot y^{oj}$ implies $y \notin Y^j$.
- $\sum_{i\in H} w^{0i} \leq \sum_{j\in F} y^{0j} + r$.
- For each $i \in H$, by $p^0 \bullet w^{0i} = M^i(p^0) = p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{0j})$, and summing over

households,

$$\sum_{i\in H} p^0 \bullet w^{0i} = \sum_i M^i(p^0) = \sum_i \left[p^0 \cdot r^i + \sum_j \alpha^{ij}(p^0 \cdot y^{oj}) \right]$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_i \sum_j \alpha^{ij} y^{0j}$$

$$= p^0 \cdot \sum_i r^i + p^0 \cdot \sum_j \sum_i \alpha^{ij} y^{0j}$$

$$= p^0 \cdot r + p^0 \cdot \sum_j y^{0j} \quad \text{(since for each j, } \sum_i \alpha^{ij} = 1 \text{).}$$

Proof by contradiction. Suppose, contrary to the theorem, there is an attainable allocation v^i , $i \in H$, so that $u^i(v^i) \geq u^i(w^{0i})$ all i with $u^h(v^h)>u^h(w^{0h})$ for some $h \in H$. The allocation v^i must be more expensive than w^{0i} for those households made better off and no less expensive for the others. Then we have

$$\sum_{i\in H} p^0 \cdot v^i > \sum_{i\in H} p^0 \cdot w^{0i} = \sum_{i\in H} M^i(p^0) = p^0 \cdot r + p^0 \cdot \sum_{j\in F} y^{0j} \ .$$

But if v^i is attainable, then there is $y'^j \in Y^j$ for each $j \in F$, so that

$\sum_{i\in H} v^i \leq \sum_{j\in F} y'^j + r$, where the inequality holds co-ordinatewise. But then, evaluating this production plan at the equilibrium prices, p^o, we have

$$p^0 \cdot r + p^0 \cdot \sum_{j\in F} y^{0j} < p^0 \cdot \sum_{i\in H} v^i \leq p^0 \cdot \sum_{j\in F} y'^j + p^0 \cdot r \ .$$

So $p^0 \cdot \sum_{j\in F} y^{0j} < p^0 \cdot \sum_{j\in F} y'^j$. Therefore for some $j \in F$, $p^0 \cdot y^{0j} < p^0 \cdot y'^j$.

But y^{0j} maximizes $p^0 \cdot y$ for all $y \in Y^j$; there cannot be $y'^j \in Y^j$ so that

$p\cdot y'^{j} > p\cdot y^{0j}$. This is a contradiction. Hence, $y'^{j} \notin Y^{j}$. The contradiction shows that v^i is not attainable. Q.E.D.

1FTWE does not require convexity.

12.3 Second Fundamental Theorem of Welfare Economics (2FTWE)

Separating Hyperplane Theorem

The Separating Hyperplane Theorem says that if we have two disjoint convex sets in R^N we can find a (hyper)plane between them so that one of the two sets is above the plane and the other below. The plane separates the convex sets. Since the plane is linear, it is defined by an equation that looks like a price system for N commodities.

Let $p \in R^N$, $p \neq 0$. Then we define a hyperplane with normal p and constant k to be a set of the form $H \equiv \{x | x \in R^N,\ p\cdot x = k\}$, where k is a real number. Note that for any two vectors x and y in H, $p\cdot(x - y) = 0$.

H divides R^N into two subsets, the portion "above" H, and "below" as measured by the dot product of p with points of R^N. The closed half space above H, is defined as the set $\{x | x \in R^N,\ p\cdot x \geq k\}$. The closed half space below H is defined as $\{x | x \in R^N,\ p\cdot x \leq k\}$.

Theorem 2.12, Separating Hyperplane Theorem: Let $A, B \subset R^N$; let A and B be nonempty, convex, and disjoint, that is $A \cap B = \phi$. Then there is $p \in R^N$, $p \neq 0$, so that $p \bullet x \geq p \bullet y$, for all $x \in A$, $y \in B$.

Let $A^i(x^i) \equiv \{x | x \in X^i,\ u^i(x) \geq u^i(x^i)\}$.

Theorem 12. 2: Assume P.I-P.IV and C.I-C.VI. Let x^{*i}, y^{*j}, $i \in H$, $j \in F$, be an attainable Pareto efficient allocation. Then there is $p \in P$ so that

(i) x^{*i} minimizes $p\cdot x$ on $A^i(x^{*i})$, $i \in H$, and

(ii) y^{*j} maximizes $p\cdot y$ on Y^j, $j \in F$.

Proof: Let $x^* = \sum_{i\in H} x^{*i}$, and let $y^* = \sum_{j\in F} y^{*j}$. Note that $x^* \leq y^* + r$ (the inequality applies co-ordinatewise). Let $A = \sum_{i\in H} A^i(x^{*i})$. Let $B = Y + \{r\}$. A and B are closed convex sets with common points, x^*, $y^* + r$. Let $\mathcal{A} = \sum_{i\in H} \{x \mid x \in X^i, u^i(x) > u^i(x^{*i})\}$. A = closure ($\mathcal{A}$).

$\mathcal{A}$ and B are disjoint, convex. By the Separating Hyperplane Theorem, there is a normal p , so that

$$p\bullet x \geq p\bullet v \quad \text{for all } x \in \mathcal{A}, \text{ and all } v \in B.$$

By continuity of u^i ,all i, and continuity of the dot product we have also $p\bullet x \geq p\bullet v$ for all $x \in A$, and all $v \in B$. x^*, $(y^*+r) \in A \cap B$, so x^* and (y^*+r) minimize $p\bullet x$ on A and maximize $p\bullet x$ on B.

By C.IV, p will be nonnegative, co-ordinatewise. Without loss of generality, let $p \in P$. x^* , $(y^*+r) \in$ A and B $\Rightarrow$ x^* and $(y^* + r)$ minimize $p\bullet x$ on A and maximize $p\bullet v$ on B. Then x^{*i} minimizes $p\bullet x$ on $A^i(x^{*i})$and y^{*j} maximizes $p\bullet y$ on Y^j. That is,

$p \cdot x^* = \min_{x \in A} p \cdot x = \min_{x^i \in A^i(x^{*i})} p \cdot \sum_{i \in H} x^i = \sum_{i \in H} (\min_{x \in A^i(x^{*i})} p \cdot x)$, and

$p \cdot (r + y^*) = \max_{v \in B} p \cdot v = p \cdot r + \sum_{j \in F} (\max_{y^j \in Y^j} p \cdot y^j)$. So x^{*i} minimizes $p \cdot x$ for all $x \in A_i(x^{*i})$ and y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$. QED

Corollary (Second Fundamental Theorem of Welfare Economics): Assume P.I-P.IV, and C.I-C.VI. Let x^{*i} , y^{*j} be an attainable Pareto efficient allocation. Then there is $p \in P$ and a choice $\hat{r}^i \geq 0$, $\hat{\alpha}^{ij} \geq 0$ so that

$$\sum_{i \in H} \hat{r}^i = r$$

$$\sum_{i \in H} \hat{\alpha}^{ij} = 1 \text{ for each j, and}$$

$$p \cdot y^{*j} \text{ maximizes } p \cdot y \text{ for } y \in Y^j$$

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \alpha^{ij}(p \cdot y^{*j})$$

and (Case 1, $p \cdot x^{*i} > \min_{x \in X^i} p \cdot x$) $u^i(x^{*i}) \geq u^i(x)$ for all $x \in X^i$ so that

$$p \cdot x \leq p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij}(p \cdot y^{*j}) ,$$

or (Case 2, $p \cdot x^{*i} = \min_{x \in X^i} p \cdot x$) x^{*i} minimizes $p \cdot x$ for all x so that $u^i(x) \geq u^i(x^{*i})$.

Proof: By Theorem 12.2, there is $p \in P$ so that y^{*j} maximizes $p \cdot y$ for all $y \in Y^j$, and so that x^{*i} minimizes $p \cdot x$ for all $x \in A^i(x^{*i})$.

By attainability, $\sum_{i \in H} x^{*i} \leq \sum_{j \in F} y^{*j} + r$. Multiplying through by p, with the recognition of free goods, we have

$$\sum_{i \in H} p \cdot x^{*i} = \sum_{j \in F} p \cdot y^{*j} + p \cdot r$$

Let $\lambda_i = \dfrac{p \cdot x^{*i}}{\sum_{h \in H} p \cdot x^{*h}}$, and set

$\hat{r}^i = \lambda_i r$, $\hat{\alpha}^{ij} = \lambda_i$, for all $i \in H$, $j \in F$.

$$p \cdot x^{*i} = p \cdot \hat{r}^i + \sum_{j \in F} \hat{\alpha}^{ij} p \cdot y^{*j}$$

Now show that cost minimization subject to utility constraint is equivalent to utility maximization subject to a budget constraint (in case 1). This follows from continuity of u^i. Suppose, on the contrary, there is x''^i so that $p \cdot x''^i = p \cdot x^{*i}$ and $u^i(x''^i) > u^i(x^{*i})$. By continuity of u^i, C.V, there is an ε neighborhood about x''^i so that all points in the neighborhood have higher utility than x^{*i} . But then some points of the neighborhood are less expensive at p than x^{*i} , and x^{*i} is no longer a cost minimizer for $A_i(x^{*i})$. This is a contradiction, hence there can be no such x''^i.

The assertion for case 2 is merely a restatement of the property shown in Theorem 2.

QED